Exercise 19

Evaluate the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$, where C is given by the vector function $\mathbf{r}(t)$.

$$\mathbf{F}(x, y) = xy^2\,\mathbf{i} - x^2\,\mathbf{j},$$
$$\mathbf{r}(t) = t^3\,\mathbf{i} + t^2\,\mathbf{j}, \quad 0 \le t \le 1$$

Solution

With the given parameterization in t, the line integral becomes

$$\begin{aligned}
\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_C \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\,dt \\
&= \int_0^1 \langle x(t)[y(t)]^2, -[x(t)]^2 \rangle \cdot \frac{d}{dt}\langle t^3, t^2 \rangle\,dt \\
&= \int_0^1 \langle t^3(t^2)^2, -(t^3)^2 \rangle \cdot \langle 3t^2, 2t \rangle\,dt \\
&= \int_0^1 \langle t^7, -t^6 \rangle \cdot \langle 3t^2, 2t \rangle\,dt \\
&= \int_0^1 [t^7(3t^2) - t^6(2t)]\,dt \\
&= \int_0^1 (3t^9 - 2t^7)\,dt \\
&= \left(\frac{3}{10}t^{10} - \frac{1}{4}t^8\right)\Big|_0^1 \\
&= \frac{3}{10} - \frac{1}{4} \\
&= \frac{1}{20}.
\end{aligned}$$